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The numerical range of elementary operators II

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Abstract

For $A, B \in L(H)$ (the algebra of all bounded linear operators on the Hilbert space H), it is proved that: (i) the generalized derivation $\delta_{A,B}$ is convexoid if and only if A and B are convexoid; (ii) the operators $\delta_{A,B}$ and $\delta_{A,B}|\mathscr{C}_p$ (where $p \geqslant 1$) have the same numerical range and are equal to $W_0(A) - W_0(B)$ (where $\mathscr{C}_p$ is the Banach space of the p-Schatten class operators on H). © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $L(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space H. All operators considered here acting on H are in $L(H)$.

If Ω is a unital complex Banach algebra and $A \in \Omega$, then we design by $\sigma(A)$, $r(A)$ and $W_0(A)$, respectively, the spectrum, the spectral radius and the numerical range of A.

For $p \geqslant 1$, we design by $(\mathscr{C}_p(H), \|.\|_p)$ the Banach space of the p-Schatten class operators on H.

We denote by tr, the trace map on $\mathscr{C}_1(H)$.

For $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ be two n-tuples of operators on H and $p \geqslant 1$, we define:

(1) the elementary operator $R(A, B)$ on $L(H)$ by

$$R(A, B)(X) = \sum_{i=1}^{n} A_i X B_i,$$

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(2) the elementary operator $R_p(A, B)$ on $\mathscr{C}_p(H)$ by

$$R_p(A, B)(X) = \sum_{i=1}^{n} A_i X B_i,$$

(3) the joint spacial numerical range $W(A)$ of A by

$$W(A) = \{(\langle A_1 x, x\rangle, \ldots, \langle A_n x, x\rangle) : \|x\| = 1\},$$

(4) the subset $W(A) \circ W(B)$ of C by

$$W(A) \circ W(B) = \left\{\sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, \ldots, \alpha_n) \in W(A), (\beta_1, \ldots, \beta_n) \in W(B)\right\}.$$

For $A, B \in L(H)$, we define the particular elementary operator on $L(H)$:

(5) the left multiplication operator by

$$\forall X \in L(H): \quad L_A(X) = AX,$$

(6) the right multiplication operator by

$$\forall X \in L(H): \quad R_B(X) = XB,$$

(7) the generalized derivation $\delta_{A,B} = L_A - R_B$ induced by A, B.

(8) the elementary multiplication operator $\mathscr{M}(A, B) = L_A R_B$ induced by A, B.

For $A, B \in L(H)$ and $p \geqslant 1$, we can also define the particular elementary operators $L_A|\mathscr{C}_p$, $R_B|\mathscr{C}_p$, $\delta_{A,B}|\mathscr{C}_p$ and $\mathscr{M}p(A, B)$ on $\mathscr{C}_p(H)$ by

$$\begin{cases} \left(L_A|\mathscr{C}_p\right)(X) = L_A(X), \\ (R_B|\mathscr{C}_p)(X) = R_B(X), \\ (\delta_{A,B}|\mathscr{C}_p)(X) = \delta_{A,B}(X), \\ \mathscr{M}_p(A, B)(X) = \mathscr{M}(A, B)(X). \end{cases}$$

For $x, y \in L(H)$, we define the operator $(x \otimes y)$ on H by

$$\forall z \in H: \quad (x \otimes y)(z) = \langle z, y\rangle\, x.$$

If $\Gamma \subset$ C, we denote by Γ^- the closure of Γ and by co Γ the convex hull of Γ.

In [1, Proposition 2.2], Bouali and Charles proved that if for $A, B \in L(H)$ such that $\|A - \lambda\| = r(A - \lambda)$, $\|B - \lambda\| = r(B - \lambda)$ for all complex λ, then $\delta_{A,B}$ is convexoid.

In [3], it is proved that:

(i) For A, B be two n-tuples of operators on H: $\mathrm{co}(W(A) \circ W(B))^- \subset W_0(R(A, B))$.

(ii) For $A, B \in L(H)$: $W_0(\delta_{A,B}) = W_0(A) - W_0(B)$.

The reader will find in the second part of this paper a reformulation of known results concerning the numerical range of these operators.

In the third part, we prove that $\delta_{A,B}$ is convexoid if and only if A and B are convexoid.

In the fourth part, we prove also that results (i) and (ii) stay true, if we replace $R(A, B)$ and $\delta_{A,B}$, respectively, by $R_p(A, B)$ and $\delta_{A,B}|\mathscr{C}_p$ for all $p \geqslant 1$.

2. Preliminaries

Definition 2.1. Let Ω be a complex Banach algebra with identity I.
(1) The set of states on Ω is by definition

$$P(\Omega) = \left\{ f \in \Omega^* : f(I) = \|f\| = 1 \right\}.$$

(2) The numerical range of an element A in Ω is by definition the set

$$W_0(A) = \left\{ f(A) : f \in P(\Omega) \right\}.$$

(3) An element A in Ω is called convexoid if $W_0(A) = \mathrm{co}\,\sigma(A)$.
(4) The usual numerical range of $A \in L(H)$ is by the definition the set

$$W(A) = \left\{ \langle Ax, x \rangle : \|x\| = 1 \right\}.$$

Theorem 2.1 [4, Theorem 1]. *If $A \in \Omega$, then $W_0(A)$ is a convex compact set and contains $\sigma(A)$.*

Theorem 2.2 [4, Theorem 6]. *If $A \in L(H)$, then $W_0(A) = W(A)^-$.*

Theorem 2.3 [5]. *If $A \in \Omega$ and $\|A - \lambda\| = r(A - \lambda)$ for all complex λ, then A is convexoid.*

Theorem 2.4 [3, Theorem 1]. *If A, B are two n-tuples of operators on H, then* $\mathrm{co}(W(A) \circ W(B))^- \subset W_0(R(A, B))$.

Theorem 2.5 [3, Theorem 2]. *If $A, B \in L(H)$, then $W_0(\delta_{A,B}) = W_0(A) - W_0(B)$.*

3. The convexoid generalized derivation

Theorem 3.1. *Let $A, B \in L(H)$. Then $\delta_{A,B}$ is convexoid if and only if A and B are convexoid.*

The proof of this theorem results from the following lemmas.

Lemma 3.1. *Let M, N and K be three convex compact subsets of C.*
(i) *If $M + N \subset M + K$, then $N \subset K$.*
(ii) *If $M + N = M + K$, then $N = K$.*

Proof. (i) Let a in N and choose b_1 in M. Then there exist b_2 in M and c_1 in K such that $a + b_1 = b_2 + c_1$. By the same, since $a + b_2 \in M + K$, then we can also choose b_3 in M and c_2 in K such that $a + b_2 = b_3 + c_2$. Then by induction, we can construct a sequence (b_n) in M and a sequence (c_n) in K such that

$$a + b_n = b_{n+1} + c_n, \quad n \geqslant 1.$$

So we obtain

$$na + b_1 = (c_1 + \cdots + c_n) + b_{n+1}, \quad n \geqslant 1,$$

and also

$$a = \frac{1}{n}(c_1 + \cdots + c_n) + \frac{1}{n}(b_{n+1} - b_1), \quad n \geqslant 1.$$

Since K is convex and M is bounded,

$$\frac{1}{n}(c_1 + \cdots + c_n) \in K \quad \text{and} \quad \lim \frac{1}{n}(b_{n+1} - b_1) = 0.$$

It follows that $a \in K$, since K is closed.

(ii) Follows immediately from (i). □

Lemma 3.2. *Let M, N, K and L be four convex compact subsets of C. If $M + N = K + L$ and $M \subset K$, $N \subset L$, then $M = K$ and $N = L$.*

Proof. Since $K + N \subset M + N = K + L \subset K + N$, then $K + L = K + N$.

It follows from Lemma 3.1, $N = L$; and by the same, $M = K$. □

Proof of Theorem 3.1. Assume that $W_0(\delta_{A,B}) = \operatorname{co}\sigma(\delta_{A,B})$; since $W_0(\delta_{A,B}) = W_0(A) - W_0(B)$, by Theorem 2.5 and since $\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$, by [2, Corollary 3.20], we have

$$W_0(A) - W_0(B) = \operatorname{co}(\sigma(A) - \sigma(B)) = \operatorname{co}\sigma(A) - \operatorname{co}\sigma(B),$$

and since $\operatorname{co}\sigma(A)$, $\operatorname{co}\sigma(B)$, $W_0(A)$ and $W_0(B)$ are convex compact with $\operatorname{co}\sigma(A) \subset W_0(A)$, $\operatorname{co}\sigma(B) \subset W_0(B)$, then by Lemma 3.2, we obtain $W_0(A) = \operatorname{co}\sigma(A)$ and $W_0(B) = \operatorname{co}\sigma(B)$.

Now, if A and B are convexoid, it follows that

$$\begin{aligned} W_0(\delta_{A,B}) &= W_0(A) - W_0(B) \\ &= \operatorname{co}\sigma(A) - \operatorname{co}\sigma(B) \\ &= \operatorname{co}(\sigma(A) - \sigma(B)) \\ &= \operatorname{co}(\sigma(\delta_{A,B})). \end{aligned} \qquad \square$$

Remark 3.1. Theorem 3.1 gives a characterization of a convexoid generalized derivation, and also it is a generalization of [1, Proposition 2.2], by using Theorem 2.3. Note that Theorem 3.1 is false if we replace the generalized derivation $\delta_{A,B}$ by the elementary multiplication operator $\mathscr{M}(A, B)$.

Indeed, if A, B are two nonscalar self-adjoint operators, then, by [3, Theorem 3], $W_0(\mathscr{M}(A, B))$ is not real, but $\operatorname{co}\sigma(\mathscr{M}(A, B)) = \operatorname{co}(\sigma(A) \cdot \sigma(B))$ is.

4. The numerical range of elementary operator acting on $\mathscr{C}_p(H)$

Theorem 4.1. *Let A and B be two n-tuples of operators on H and let $p \geqslant 1$. Then* $\mathrm{co}(W(A) \circ W(B))^- \subset W_0(R_p(A, B))$.

Proof. Let $x, y \in H$ such that $\|x\| = \|y\| = 1$.

Define the map f on $L(\mathscr{C}_p(H))$ by

$$\forall F \in L(\mathscr{C}_p(H))\colon\ f(F) = \mathrm{tr}\left[(y \otimes x)F(x \otimes y)\right].$$

Since $\|x \otimes y\|_p = \|x \otimes y\|_1 = \|x \otimes y\| = \|x\|\|y\| = 1$, and since $\|X\| \leqslant \|X\|_p$ for all $X \in \mathscr{C}_p(H)$, we have

$$\begin{aligned}\left|f(F)\right| &\leqslant \|(y \otimes x)F(x \otimes y)\|_1 \\ &\leqslant \|x \otimes y\|_1 \,\|F(x \otimes y)\| \\ &\leqslant \|F(x \otimes y)\|_p \\ &\leqslant \|F\|\end{aligned}$$

and $f(I) = 1$ so that f is a state on $L(\mathscr{C}_p(H))$; and since

$$f(R_p(A, B)) = \sum_{i=1}^{n} \langle A_i x, x\rangle \cdot \langle B_i y, y\rangle \in W_0(R_p(A, B)),$$

we obtain $W(A) \circ W(B) \subset W_0(R_p(A, B))$, and since $W_0(R_p(A, B))$ is compact and convex, thus $\mathrm{co}(W(A) \circ W(B))^- \subset W_0(R_p(A, B))$. $\square$

Corollary 4.1. *Let $A \in L(H)$. Then $W_0(L_A|\mathscr{C}_p) = W_0(R_A|\mathscr{C}_p) = W_0(A)$.*

Proof. The inclusions $W_0(A) \subset W_0(L_A|\mathscr{C}_p)$, $W_0(A) \subset W_0(R_A|\mathscr{C}_p)$ follow immediately from Theorems 4.1 and 2.2.

Now, let f be a state on $L(\mathscr{C}_p(H))$ and we define the map g on $L(H)$ by $g(X) = f(L_X|\mathscr{C}_p)$. By a simple computation, we find that g is a state on $L(H)$ so that $g(A) = f(L_A|\mathscr{C}_p) \in W_0(A)$. Therefore $W_0(L_A|\mathscr{C}_p) \subset W_0(A)$. By the same, we find also $W_0(R_A|\mathscr{C}_p) \subset W_0(A)$. $\square$

Corollary 4.2. *Let $A, B \in L(H)$ and $p \geqslant 1$. Then $W_0(\delta_{A,B}|\mathscr{C}_p) = W_0(\delta_{A,B})$.*

Proof. By Theorems 4.1, 2.2 and Corollary 4.1, we obtain

$$\begin{aligned}W_0(A) - W_0(B) &\subset W_0(\delta_{A,B}|\mathscr{C}_p) \subset W_0(L_A|\mathscr{C}_p) - W_0(R_B|\mathscr{C}_p) \\ &= W_0(A) - W_0(B). \qquad \square\end{aligned}$$

Remark 4.1. Corollary 4.2 is false if we replace the generalized derivation $\delta_{A,B}$ by the elementary multiplication operator $\mathscr{M}(A, B)$.

Indeed, if A, B are two nonscalar self-adjoint operators and $p=2$, then $W_0(\mathscr{M}(A, B))$ is not real but $W_0(\mathscr{M}_2(A, B))$ is, because $\mathscr{M}_2(A, B)$ is a self-adjoint operator on the Hilbert space $\mathscr{C}_2(H)$.

Corollary 4.3. *Let $A, B \in L(H)$ and $p \geqslant 1$. Then $\delta_{A,B}|\mathscr{C}_p$ is convexoid if and only if A and B are convexoid.*

Proof. Since $W_0(\delta_{A,B}|\mathscr{C}_p) = W_0(\delta_{A,B})$ and $\sigma(\delta_{A,B}|\mathscr{C}_p) = \sigma(\delta_{A,B})$, the proof follows immediately if we use Theorem 3.1. $\square$

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